Distribution of the time to buffer overflow in the single-server queueing model with multiple vacation policy

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Abstract. In this paper we consider a single-server queueing model with finite buffer capacity and the multiple vacation policy, in which jobs occur according to a Poisson process and are being processed individually with a general-type cumulative distribution function of the service time. Every time when the system empties, a service station initializes a multiple vacation period. During this period successive generally-distributed vacations are being started one by one until at least one packet accumulated in the buffer is detected at the completion epoch of one of them. A compact formula for the Laplace transform of the distribution of the time to the first buffer overflow is found. The result is written using a recurrent sequence, defined by means of "input" characteristics of the system. Numerical examples are attached as well.

Keywords: buffer overflow, finite buffer, multiple vacation policy, Poisson process, queueing system.

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1. Motivation

Obviously, a phenomenon of packet (job) losses as a natural consequence of buffer saturation is a typical one for packet-oriented computer and telecommunication networks. The more in-depth investigation of this process requires knowledge of some stochastic characteristics, like e.g. distributions of successive buffer overflow periods and times of reaching them. The loss ratio, defined as the part of the total number of transmitted packets, which are lost due to the buffer saturation, is not sufficient here.

The review of steady-state results for finite-buffer queueing models can be found e.g. in [7, 8] and [17]. In [11] distributions of three different characteristics for the system with the arrival stream of packets "filtered" by an active queue management

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(AQM for short) algorithm are derived. The time-dependent (transient) analysis of a finite-buffer queue can be found e.g. in [12], where the departure counting process in the system with batch Poisson arrivals is considered.

One can find analytical results for the cumulative distribution function (CDF for short) of the time to the buffer overflow in single-server queues without any additional policy limiting the access to the service station e.g. in [4, 5] and [6]. In particular, in [4] the system with batch Poisson arrivals and constant processing times is analyzed. In [5] the closed-form representation for the distribution of the time to the first buffer overflow is obtained in the model with BMAP-type input stream. Results for the MMPP-type arrival flow can be found in [6]. Some other results for the loss process and buffer saturation problem are given e.g. in [10] and [15]. In [10] the representation for the joint transform of the busy period and numbers of packets being processed and lost during the busy period is derived for the model with phase-type key “input” distributions. In [15] the formula for the distribution of the number of buffer overflows during one busy period is found for Markovian packet arrivals.

In this paper we deal with CDF of the time to the first buffer overflow in a finite-buffer model operating under the multiple vacation policy. Applying analytical approach based on the idea of embedded Markov chain, continuous total probability law, integral equations and linear algebra, we obtained the explicit closed-form representation for the Laplace transform (LT for short) of the CDF of the time to the first buffer overflow, conditioned by the number of packets accumulated in the buffer at the starting moment.

The article is organized as follows. In Section 2 we give a precise mathematical description of the considered queueing model and state some auxiliary results. In Section 3 we present main analytical results for the CDF of the time to the first buffer overflow. Section 4 contains some numerical examples and in the last Section 5 we give conclusions and final remarks.

2. Preliminaries

We consider an $M/G/1/N$-type queueing system in which packets enter according to a Poisson process with rate λ and are being processed individually, with a generally-distributed service time with CDF $F(\cdot)$ with Laplace-Stieltjes transform (LST for short) $f(\cdot)$. The FIFO service discipline is assumed. The number of packets permitted to be simultaneously present in the system is bounded by a non-random value N, so we have $N-1$ places in the buffer queue and one place in processing. Every time when the system empties, the service station initializes a multiple vacation period, during which successive vacations are being started one by one until at least one packet accumulated in the buffer is detected at one of them. We assume that all vacations are i.i.d. (=independent and identically distributed) random variables with common general-type CDF $V(\cdot)$ with LST $v(\cdot)$.

Let $X(t)$ be the number of packets present in the system at the time t. Denote by β_n the time to the first overflow of the buffer under condition that the buffer initially (at the starting time $t=0$) contains exactly n packets. In other words, we can define β_n as

$$\beta_n = \inf\{t > 0 : X(t) = N \,|\, X(0) = n\}, \tag{1}$$

where $0 \leq n \leq N - 1$.

The approach we propose for the study of the CDF of the time to buffer overflow leads to a system of linear equations. To find the solution in a compact form we utilize the following result [13] (see also [14]):

Theorem 2.1. *Let $(a_k)_{k=0}^{\infty}$, $a_0 \neq 0$, and $(\phi_k)_{k=1}^{\infty}$ be sequences.*

Each solution of the following system of equations:

$$\sum_{k=-1}^{n-1} a_{k+1} x_{n-k} - x_n = \phi_n, \quad n \geq 1, \tag{2}$$

can be written in the following form:

$$x_n = CR_n + \sum_{k=1}^{n} R_{n-k}\phi_k, \quad n \geq 1, \tag{3}$$

where C is a certain constant independent on n and (R_k), $k = 0, 1, ...,$ is a specific-type sequence connected with the given sequence (a_k) and defined recursively in the following way:

$$R_0 = 0, \quad R_1 = a_0^{-1}, \quad R_{k+1} = R_1(R_k - \sum_{i=0}^{k} a_{i+1} R_{k-i}), \quad k \geq 1. \tag{4}$$

3. Analytical results

3.1. Integral equations for conditional CDF of time to buffer overflow

Let

$$\Delta_n(t) \overset{def}{=} \mathbf{P}\{\beta_n > t\}, \quad t \geq 0,\, 0 \leq n \leq N - 1, \tag{5}$$

i.e. $\Delta_n(\cdot)$ is the tail of the conditional CDF of the time to the first buffer overflow.

Assume firstly that the system is empty ($n = 0$) at the starting epoch $t = 0$. In such a case the multiple vacation period begins at this time. Let us note that then one can distinguish three mutually separable random events:

- the first packet enters before time t and the multiple vacation period also ends before t (we denote this event by $L_1(t)$);
- the first packet enters before time t and the multiple vacation period ends after t (we denote this event by $L_2(t)$);
- the first packet arrives after time t (we denote this event by $L_3(t)$).

Evidently, we have

$$\Delta_0(t) = \sum_{i=1}^{3} \Delta_{0,i}(t), \tag{6}$$

where

$$\Delta_{0,i}(t) = \mathbf{P}\{\beta_0 > t\,,\, L_i(t)\}, \;\; i = 1, 2, 3. \tag{7}$$

Observe that for successive functionals $\Delta_{0,i}(t)$ the following formulae can be derived:

$$\Delta_{0,1}(t) = \lambda \int_{x=0}^{t} e^{-\lambda x} dx \sum_{i=0}^{\infty} \int_{y=0}^{x} dV^{i*}(y) \int_{u=x-y}^{t-y} \sum_{k=0}^{N-2} \frac{\big[\lambda(u+y-x)\big]^k}{k!} \times$$
$$\times\, e^{-\lambda(u+y-x)} \Delta_{k+1}(t-u-y) dV(u), \tag{8}$$
$$\Delta_{0,2}(t) = \lambda \int_{x=0}^{t} e^{-\lambda x} dx \sum_{i=0}^{\infty} \int_{y=0}^{x} \overline{V}(t-y) \sum_{k=0}^{N-2} \frac{\big[\lambda(t-x)\big]^k}{k!} e^{-\lambda(t-x)} dV^{i*}(y) \tag{9}$$

and

$$\Delta_{0,3}(t) = e^{-\lambda t}, \tag{10}$$

where $\overline{V}(x) \overset{def}{=} 1 - V(x)$.

Hence taking (6) under consideration, we get

$$\Delta_0(t) = \lambda \int_{x=0}^{t} e^{-\lambda x} dx \sum_{i=0}^{\infty} \int_{y=0}^{x} dV^{i*}(y) \int_{u=x-y}^{t-y} \sum_{k=0}^{N-2} \frac{\big[\lambda(u+y-x)\big]^k}{k!}$$
$$\times\, e^{-\lambda(u+y-x)} \Delta_{k+1}(t-u-y) dV(u)$$
$$+\, \lambda \int_{x=0}^{t} e^{-\lambda x} dx \sum_{i=0}^{\infty} \int_{y=0}^{x} \overline{V}(t-y) \sum_{k=0}^{N-2} \frac{\big[\lambda(t-x)\big]^k}{k!} e^{-\lambda(t-x)} dV^{i*}(y) + e^{-\lambda t}. \tag{11}$$

Assume now that the system is not empty at the beginning ($n > 0$). Utilizing the formula of total probability with respect to the first departure epoch after the opening of the system, by the fact that departure epochs are Markov moments in the considered queueing system, we obtain the following equation:

$$\Delta_n(t) = \sum_{i=0}^{N-n-1} \int_0^t \Delta_{n+i-1}(t-y) \frac{(\lambda y)^i}{i!} e^{-\lambda y} dF(y) + \overline{F}(t) \sum_{i=0}^{N-n-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t}, \tag{12}$$

where $1 \le n \le N-1$ and $\overline{F}(t) \overset{def}{=} 1 - F(t)$. The first summand on the right side of (12) relates to the situation in which the first departure occurs before time t, while the second one – to the opposite case.

3.2. The corresponding system for LTs

Define

$$\delta_j(s) \overset{def}{=} \int_0^\infty e^{-st}\Delta_j(t)dt, \quad \mathrm{Re}(s) > 0. \tag{13}$$

Observe that the following representations hold (compare to (8)):

$$\begin{aligned}
&\frac{\lambda^{k+1}}{k!}\int_{t=0}^\infty e^{-st}dt\int_{y=0}^t dV^{i*}(y)\int_{x=y}^t e^{-\lambda x}dx\int_{u=x-y}^{t-y}(u+y-x)^k e^{-\lambda(u+y-x)} \\
&\times \Delta_{k+1}(t-u-y)dV(u) = \frac{\lambda^{k+1}}{k!}\int_{t=0}^\infty e^{-st}dt\int_{y=0}^t e^{-\lambda y}dV^{i*}(y)\int_{u=0}^{t-y} e^{-\lambda u} \\
&\times \Delta_{k+1}(t-u-y)dV(u)\int_{x=y}^{u+y}(u+y-x)^k dx \\
&= \frac{\lambda^{k+1}}{(k+1)!}\int_{t=0}^\infty e^{-st}dt\int_{y=0}^t e^{-\lambda y}dV^{i*}(y)\int_{u=0}^{t-y} e^{-\lambda u}u^{k+1}\Delta_{k+1}(t-u-y)dV(u) \\
&= \frac{\lambda^{k+1}}{(k+1)!}\int_{y=0}^\infty e^{-(\lambda+s)y}dV^{i*}(y)\int_{u=0}^\infty e^{-(\lambda+s)u}u^{k+1}dV(u) \\
&\times \int_{t=u+y}^\infty e^{-s(t-u-y)}\Delta_{k+1}(t-u-y)dt = v^i(\lambda+s)g_{k+1}(s)\delta_{k+1}(s),
\end{aligned} \tag{14}$$

where

$$g_j(s) \overset{def}{=} \frac{\lambda^j}{j!}\int_{u=0}^\infty e^{-(\lambda+s)u}u^j dV(u). \tag{15}$$

Similarly, we get (compare to (9))

$$\begin{aligned}
&\frac{\lambda^{k+1}}{k!}\int_{t=0}^\infty e^{-st}dt\int_{x=0}^t e^{-\lambda x}dx\int_{y=0}^x \overline{V}(t-y)\frac{\left[\lambda(t-x)\right]^k}{k!}e^{-\lambda(t-x)}dV^{i*}(y) \\
&\frac{\lambda^{k+1}}{k!}\int_{t=0}^\infty e^{-(\lambda+s)t}dt\int_{y=0}^t \overline{V}(t-y)dV^{i*}(y)\int_{x=y}^t (t-x)^k dx \\
&= \frac{\lambda^{k+1}}{(k+1)!}\int_{t=0}^\infty e^{-(\lambda+s)t}dt\int_{y=0}^t \overline{V}(t-y)(t-y)^{k+1}dV^{i*}(y) \\
&= \frac{\lambda^{k+1}}{(k+1)!}\int_{y=0}^\infty e^{-(\lambda+s)y}dV^{i*}(y)\int_{t=y}^\infty \overline{V}(t-y)(t-y)^{k+1}dt \\
&= v^i(\lambda+s)\overline{g}_{k+1}(s),
\end{aligned} \tag{16}$$

where

$$\overline{g}_j(s) \overset{def}{=} \frac{\lambda^j}{j!}\int_0^\infty e^{-(\lambda+s)u}u^j\overline{V}(u)du. \tag{17}$$

By (11), (14) and (16) we obtain

$$\delta_0(s) = \frac{1}{1-v(\lambda+s)} \sum_{k=0}^{N-2} \Big(g_{k+1}(s)\delta_{k+1}(s) + \overline{g}_{k+1}(s) \Big) + \frac{1}{\lambda+s}$$
$$= \frac{1}{1-v(\lambda+s)} \sum_{k=1}^{N-1} \Big(g_k(s)\delta_k(s) + \overline{g}_k(s) \Big) + \frac{1}{\lambda+s}. \tag{18}$$

If we define

$$a_j(s) \stackrel{def}{=} \int_0^\infty e^{-(\lambda+s)t} \frac{(\lambda t)^j}{j!} dF(t) \tag{19}$$

and

$$b_j(s) \stackrel{def}{=} \int_0^\infty e^{-(\lambda+s)t} \sum_{i=0}^{j} \frac{(\lambda t)^i}{i!} \overline{F}(t) dt, \tag{20}$$

we can transform (12) as follows:

$$\delta_n(s) = \sum_{i=0}^{N-n-1} a_i(s)\delta_{n+i-1}(s) + b_{N-n-1}(s), \quad 1 \le n \le N-1. \tag{21}$$

3.3. The main analytical result

If we denote

$$H_{N-n}(s) \stackrel{def}{=} \delta_n(s), \quad 0 \le n \le N-1, \tag{22}$$

then the system (21) can be rewritten as follows:

$$\sum_{i=-1}^{n-1} H_{n-i}(s)a_{i+1}(s) - H_n(s) = \phi_n(s), \;\; 1 \le n \le N-1, \tag{23}$$

where

$$\phi_n(s) = H_1(s)a_n(s) - b_{n-1}(s). \tag{24}$$

Moreover, equation (18) can be reformulated in the following way:

$$H_N(s) = \frac{1}{1-v(\lambda+s)} \sum_{k=1}^{N-1} \Big(g_k(s)H_{N-k}(s) + \overline{g}_k(s) \Big) + \frac{1}{\lambda+s}. \tag{25}$$

To obtain the solution of the system (23) and (25), we can apply Theorem 2.1 as (23) suits (2) with unknown functions $H_n(s)$ and all coefficients depending on the variable s. Hence, to find the solution we can utilize the formula (3). Moreover, since the number of equations in (23) is finite, we can use the equation (25) as a specific-type boundary condition allowing to eliminate $C = C(s)$.

Indeed, we have, firstly,

$$H_n(s) = C(s)R_n(s) + \sum_{k=1}^{n} R_{n-k}(s)\phi_k(s), \;\; n \geq 1, \tag{26}$$

where (compare to (4))

$$\begin{aligned} &R_0(s) = 0, \quad R_1(s) = a_0^{-1}(s), \\ &R_{k+1}(s) = R_1(s)\big(R_k(s) - \sum_{i=0}^{k} a_{i+1}(s)R_{k-i}(s)\big), \quad k \geq 1. \end{aligned} \tag{27}$$

For $n = 1$ from (26) we get

$$H_1(s) = C(s)R_1(s). \tag{28}$$

Similarly, taking $n = N$ in (26), by (24) and (28) we obtain

$$H_N(s) = C(s)R_N(s) + \sum_{k=1}^{N} R_{N-k}(s)\big[C(s)R_1(s)a_k(s) - b_{k-1}(s)\big]. \tag{29}$$

Simultaneously, substituting (26) into (25) yields

$$\begin{aligned} H_N(s) = &\frac{1}{1 - v(\lambda + s)} \sum_{k=1}^{N-1} \Bigg\{ g_k(s)\Big[C(s)R_{N-k}(s) \\ &+ \sum_{i=1}^{N-k} R_{N-k-i}(s)\big(C(s)R_1(s)a_i(s) - b_{i-1}(s)\big)\Big] + \overline{g}_k(s)\Bigg\} + \frac{1}{\lambda + s}. \end{aligned} \tag{30}$$

Comparing the right sides of (29) and (30) we get

$$C(s) = A(s)B(s), \tag{31}$$

where

$$\begin{aligned} A(s) \overset{def}{=} &\frac{1}{1 - v(\lambda + s)} \sum_{k=1}^{N-1} \Big(\overline{g}_k(s) - g_k(s) \sum_{i=1}^{N-k} R_{N-k-i}(s)b_{i-1}(s)\Big) \\ &+ \sum_{k=1}^{N} R_{N-k}(s)b_{k-1}(s) + \frac{1}{\lambda + s} \end{aligned} \tag{32}$$

and

$$B(s) \stackrel{def}{=} \Bigg[R_N(s) + R_1(s)\sum_{k=1}^{N} R_{N-k}(s)a_k(s) - \frac{1}{1-v(\lambda+s)}\sum_{k=1}^{N-1} g_k(s)$$
$$\times \Big(R_{N-k}(s) + R_1(s)\sum_{i=1}^{N-k} R_{N-k-i}(s)a_i(s)\Big)\Bigg]^{-1}. \tag{33}$$

From (22), (24), (26) and (31), we deduce the following:

Theorem 3.1. *The representation for LT of the tail of the CDF of the time to the first buffer overflow in the $M/G/1/N$-type queueing model with multiple vacation policy is following:*

$$\delta_n(s) = \int_0^\infty e^{-st}\Delta_n(t)dt = A(s)B(s)\Big(R_{N-n}(s) + R_1(s)\sum_{k=1}^{N-n} R_{N-n-k}(s)a_k(s)\Big)$$
$$-\sum_{k=1}^{N-n} R_{N-n-k}(s)b_{k-1}(s), \tag{34}$$

where $1 \le n \le N-1$, $\mathrm{Re}(s) > 0$ *and the formulae for* $a_k(s)$, $b_k(s)$, $R_k(s)$, $A(s)$ *and* $B(s)$ *are given in (19), (20), (27), (32) and (33), respectively.*

4. Numerical examples

In this section we present some numerical examples illustrating theoretical results, for which we discuss the dependence of the distribution of the time to the first buffer overflow on server vacation duration, intensity of arrivals, processing speed and initial buffer state. Assume that packets of average sizes 500 B arrive at the node of WSN (wireless sensor network) according to a Poisson process with rate λ and are being processed individually, according to FIFO service discipline, with exponentially distributed service time with mean μ^{-1}. For numerical calculations we also assume that a multiple vacation period consists of independent exponentially distributed server vacations, each one with mean $1/\lambda v$. First we use the formula (35) in Theorem 3.1 to obtain explicit representations for Laplace transforms of the time to the first buffer overflow. Next, we use procedures of numerical Laplace transform inversion, based on algorithms of Abate-Choudhury-Whitt presented in [1] and the Gaver-Stehfest algorithm proposed in [2, 3] (which is a combination of two approaches given in [9] and [16]). The results are illustrated in appropriate figures or tables. We get the mean value of the time to buffer overflow for fixed set of system parameters from the evident relationship, that is

$$\mathbf{E}\{\beta_n\} = \delta_n(0) = \int_0^\infty \Delta_n(t)dt. \tag{35}$$

4.1. Impact of initial buffer state

In this example we investigate the dependence of $\varDelta_n(t)$ on the initial buffer state for $n = 0, 7, 11$ and 14, where the maximum system capacity equals $N = 15$, the arrival rate $\lambda = 25 * 10^4$ packets/s (corresponding to intensity 1 Gb/s), the processing rate $\mu = 3 * 10^5$ packets/s (corresponding to intensity 1.2 Gb/s), and the server vacations are exponentially distributed with parameter $\lambda v = 6 * 10^5$. The results are presented in Fig. 1.

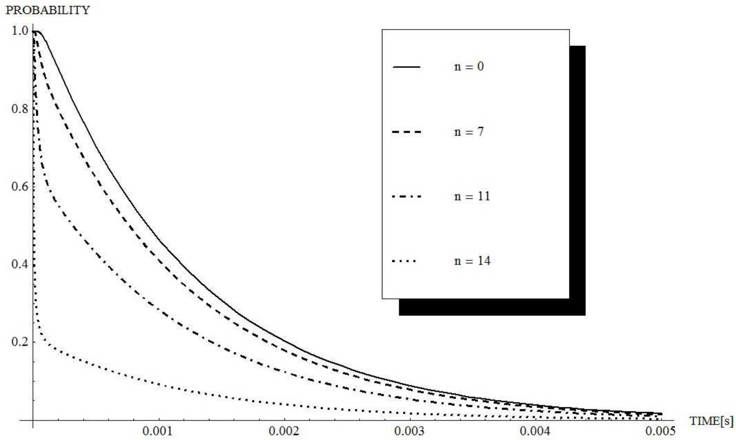

Fig. 1. $\varDelta_n(t)$ in dependence on the initial buffer state n

4.2. Impact of arrival intensity

Here we visualize the dependence of $\varDelta_n(t)$ on the intensity of packet arrivals for $\lambda = 10 * 10^4, 15 * 10^4, 20 * 10^4, 25 * 10^4$ and $30 * 10^4$ packets/s (corresponding to intensities 400 Mb/s, 600 Mb/s, 800 Mb/s, 1 Gb/s and 1.2 Gb/s, respectively), where the initial buffer state $n = 1$, the maximum system capacity $N = 15$, the processing rate $\mu = 2 * 10^5$ packets/s (corresponding to intensity 800 Mb/s), and the parameter of exponentially distributed server vacation $\lambda v = 6 * 10^5$. The results are given in Fig. 2.

4.3. Impact of processing rate

We investigate here the impact of the service speed on the time to buffer overflow for three different exponential processing rates $\mu = 20 * 10^4, 25 * 10^4$ and $30 * 10^4$ packets/s (corresponding to 800 Mb/s, 1 Gb/s and 1.2 Gb/s, respectively), where the initial buffer state $n = 2$, the maximum system capacity equals $N = 15$, the arrival rate $\lambda = 25 * 10^4$ packets/s (corresponds to intensity of 1 Gb/s), and $\lambda v = 6 * 10^5$. The results are presented in Fig. 3.

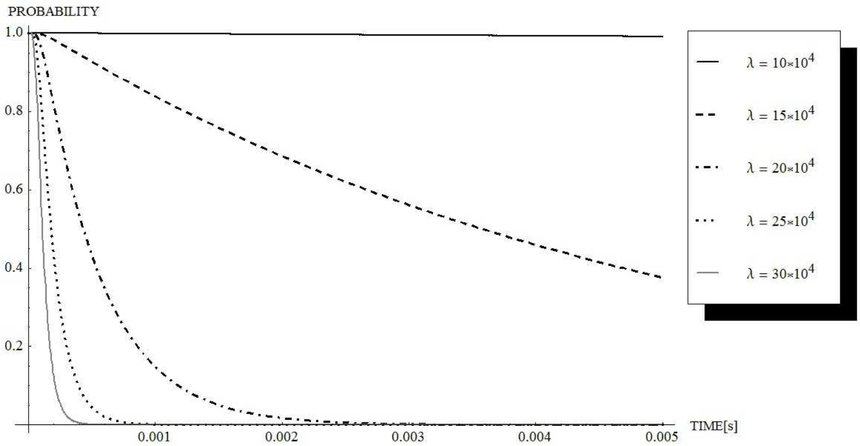

Fig. 2. $\Delta_n(t)$ in dependence on the arrival intensity λ

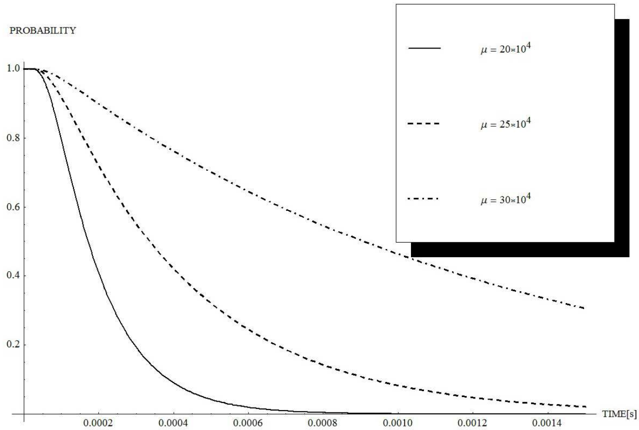

Fig. 3. $\Delta_n(t)$ in dependence on the processing rate μ

4.4. Impact of intensity of arrivals and service speed

In this example we visualize the dependence of $\Delta_n(t)$ on two factors: the arrival rate $\lambda = 15 * 10^4, 20 * 10^4$ and $25 * 10^4$ packets/s (600 Mb/s, 800 Mb/s, 1 Gb/s) and the processing rate $\mu = 20 * 10^4, 25 * 10^4$ and $30 * 10^4$ packets/s (800 Mb/s, 1 Gb/s, 1.2 Gb/s), where besides $n = 1$, $N = 5$, and $\lambda v = 6 * 10^5$. In calculations we used two different algorithms of numerical Laplace transform inversion (Abate-Choudhury-Whitt [1] and Gaver-Stehfest [2]). We also measured the time complexity for both algorithms. The average time spent on a single evaluation of Abate-Choudhury-Whitt algorithm is approximately equal to 34 seconds. It turns out that the Gaver-Stehfest algorithm is much faster, because the average time spent on single evaluation is approximately equal to 0.015 second. The results of comparison of the two methods are presented in Fig. 4.

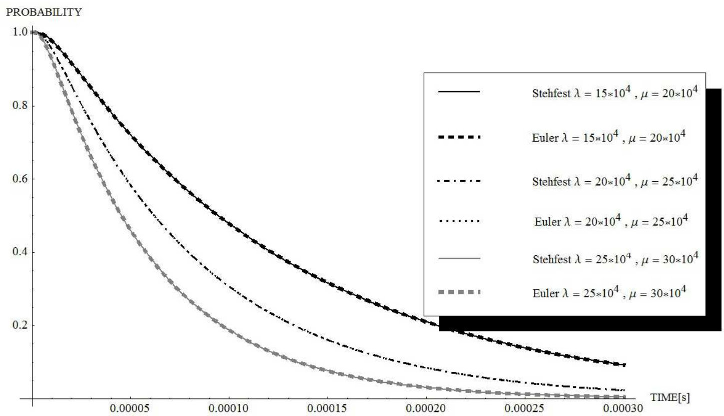

Fig. 4. $\Delta_n(t)$ in dependence on the arrival intensity λ and processing rate μ

4.5. Impact of vacation duration

In this example we visualize the dependence of the conditional CDF of the time to the first buffer overflow on the single server vacation duration, taking $\lambda v = 10 * 10^4, 15 * 10^4$ and $30 * 10^4$. The remaining system parameters are $\lambda = 10 * 10^4$ packets/s (corresponding to intensity 400 Mb/s), $n = 1$, $N = 15$, and $\mu = 25 * 10^4$ packets/s (corresponding to 1 Gb/s). The results are given in Fig. 5. In Fig. 6 we present similar results for an alternative intensity 700 Mb/s of input stream of packets.

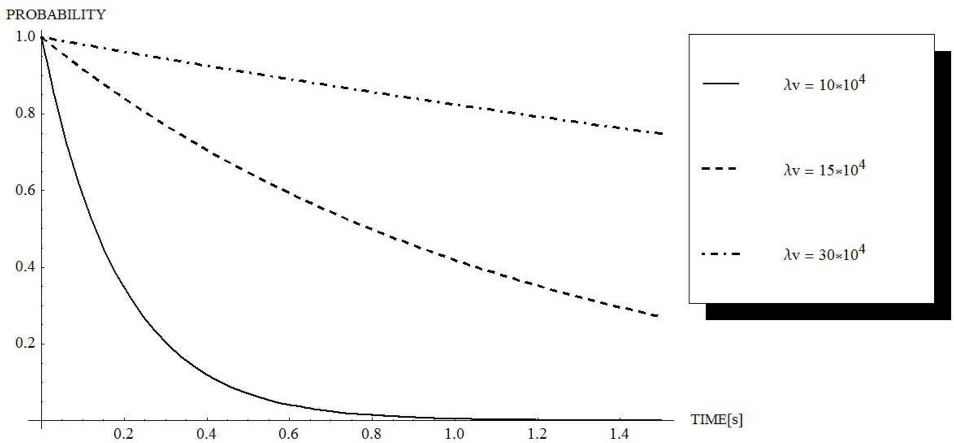

Fig. 5. $\Delta_n(t)$ in dependence on λv for input stream intensity 400 Mb/s

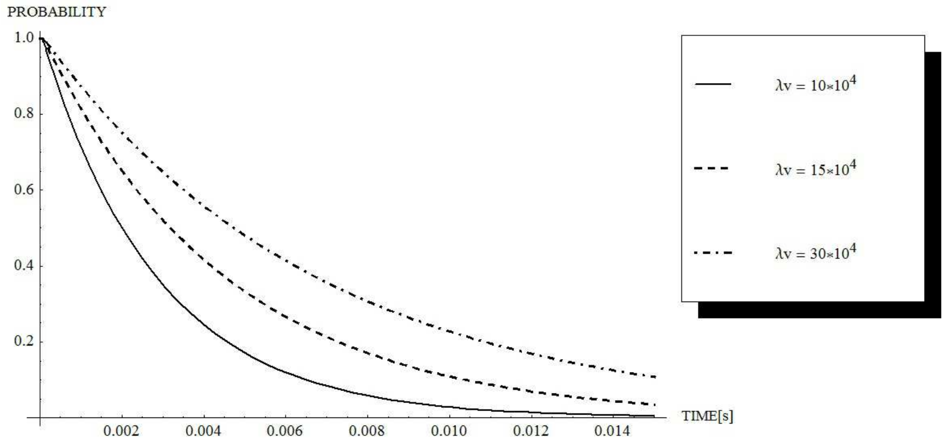

Fig. 6. $\Delta_n(t)$ in dependence on λv for input stream intensity 700 Mb/s

4.6. Impact of arrival intensity and service rate on mean time to buffer overflow

In this example we visualize in the form of a table the average time to buffer overflow for every possible initial buffer states n, in dependence on different input and output link intensities and the maximum system capacity $N = 15$, taking the server vacation parameter $\lambda v = 6 * 10^5$. The results are presented in Table 1.

4.7. Impact of initial buffer state on mean time to buffer overflow

Here we state the average time to buffer overflow for every possible initial buffer states n, in dependence on the vacation parameter $\lambda v = 10 * 10^4, 15 * 10^4$ and $30 * 10^4$, for the arrival rate $\lambda = 10 * 10^4$ packets/s (corresponding to intensity 400 Mb/s), the maximum system capacity $N = 15$ and the processing rate $\mu = 25 * 10^4$ packets/s (1 Gb/s). The results are given in Table 2. The case of intensity 700 Mb/s ($\lambda = 175 * 10^3$) is visualized in Table 3.

Table 1

Average time to buffer overflow for $N = 15$

	Average time to buffer overflow	
n	Input 800 Mb/s, Output 800 Mb/s	Input 1 Gb/s, Output 800 Mb/s
0	0.000575 sec.	0.000216 sec.
1	0.000572 sec.	0.000214 sec.
2	0.000563 sec.	0.000208 sec.
3	0.000550 sec.	0.000199 sec.
4	0.000532 sec.	0.000188 sec.
5	0.000508 sec.	0.000175 sec.
6	0.000480 sec.	0.000160 sec.
7	0.000447 sec.	0.000145 sec.
8	0.000408 sec.	0.000129 sec.
9	0.000365 sec.	0.000111 sec.
10	0.000317 sec.	0.000094 sec.
11	0.000263 sec.	0.000076 sec.
12	0.000205 sec.	0.000059 sec.
13	0.000142 sec.	0.000038 sec.
14	0.000073 sec.	0.000019 sec.

Table 2

Average time to buffer overflow for $N = 15$ and $\lambda = 10 * 10^4$ packets/s

	Average time to buffer overflow		
n	$\lambda v = 10 * 10^4$	$\lambda v = 15 * 10^4$	$\lambda v = 30 * 10^4$
0	0.188577 sec.	1.14968 sec.	5.1784 sec.
1	0.188584 sec.	1.14968 sec.	5.1784 sec.
2	0.188589 sec.	1.14969 sec.	5.17838 sec.
3	0.188594 sec.	1.14968 sec.	5.17834 sec.
4	0.188596 sec.	1.14966 sec.	5.17822 sec.
5	0.188591 sec.	1.14959 sec.	5.1779 sec.
6	0.188568 sec.	1.14942 sec.	5.17709 sec.
7	0.1885 sec.	1.14897 sec.	5.17506 sec.
8	0.188322 sec.	1.14785 sec.	5.16997 sec.
9	0.187865 sec.	1.14503 sec.	5.15725 sec.
10	0.186712 sec.	1.13797 sec.	5.12544 sec.
11	0.183821 sec.	1.12032 sec.	5.04591 sec.
12	0.176582 sec.	1.07617 sec.	4.84706 sec.
13	0.158476 sec.	0.965802 sec.	4.34993 sec.
14	0.1132 sec.	0.689861 sec.	3.1071 sec.

Table 3
Average time to buffer overflow for $N = 15$ and $\lambda = 175 * 10^3$ packets/s

	Average time to buffer overflow		
n	$\lambda v = 10 * 10^4$	$\lambda v = 15 * 10^4$	$\lambda v = 30 * 10^4$
0	0.0028589 sec.	0.00455093 sec.	0.00678827 sec.
1	0.00286598 sec.	0.00455455 sec.	0.00678732 sec.
2	0.00287038 sec.	0.00455401 sec.	0.00678024 sec.
3	0.00287095 sec.	0.00454752 sec.	0.00676442 sec.
4	0.00286605 sec.	0.00453254 sec.	0.0067361 sec.
5	0.00285334 sec.	0.00450542 sec.	0.00668993 sec.
6	0.00282947 sec.	0.00446096 sec.	0.00661826 sec.
7	0.00278965 sec.	0.00439174 sec.	0.00651016 sec.
8	0.00272704 sec.	0.00428713 sec.	0.00635002 sec.
9	0.0026319 sec.	0.00413198 sec.	0.00611552 sec.
10	0.00249026 sec.	0.00390463 sec.	0.00577482 sec.
11	0.00228221 sec.	0.00357412 sec.	0.00528239 sec.
12	0.00197928 sec.	0.00309625 sec.	0.0045732 sec.
13	0.00154081 sec.	0.00240786 sec.	0.00355435 sec.
14	0.000908714 sec.	0.00141874 sec.	0.00209315 sec.

5. Conclusions

In this paper a single-channel queueing model with finite buffer capacity operating under the multiple vacation policy is investigated. It is assumed that jobs occur according to a Poisson process and are being processed individually with a general-type CDF of the service time, according to the FIFO discipline. Each time when the system becomes empty, the service station begins a multiple vacation period, during which successive independent and generally-distributed vacations are being initialized, until the buffer is not empty at the end of one of them. By using analytical approach based on the idea of the embedded Markov chain, the formula of total probability, integral equations and linear algebra, a closed-form representation for the LT of the distribution of time to the first buffer overflow is found. The impact of different "input" system parameters, namely vacation duration, intensity of arrivals, processing rate and initial buffer state is analyzed in numerical examples.

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